Math 254B Lecture 20 Notes

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1 Billingsley's Lemma, Local Dimension, and Frostman's Lemma

1.1 Vitali's lemma

Last time we covered the mass distribution principle:

Theorem 1.1 (Mass distribution principle). *Let (X, ρ) be a metric space with $\mu \in P(X)$ and $A \in \mathcal{B}_X$. Assume that*

1. *For all $x \in A$ there is a $\delta_x > 0$ such that $\mu(B(x, \delta)) \leq C\delta^\alpha$.*
2. *$\mu(A) > 0$.*

Then $m_\alpha(A) > 0$, so $\dim_H(A) \geq \alpha$.

We want to discuss converses to this. We need the following "combinatorial" fact.

Lemma 1.1 (Vitali). *Let (X, ρ) be a compact metric space, and let $\mathcal{B} = \{B(x_i, r_i) : i \in I\}$ be a family of balls.*

1. *If I is finite, then there exist $i_1, \dots, i_n \in I$ such that $B(x_{i_j}, r_{i_j}) \cap B(x_{i_\ell}, r_{i_\ell}) = \varnothing$ for all $j \neq \ell$.*
2. *If I is general with $\sup_i r_i < \infty$, then there is a sequence $i_1, i_2, \dots \in I$ such that the $B(x_{i_j}, r_{i_j})$ are disjoint and $\bigcup \mathcal{B} \subseteq \bigcup_j B(x_{i_j}, 3.1 r_{i_j})$.*

Remark 1.1. The first case does not need the compactness. For the second case, we can substitute compactness with other assumptions.

1.2 Billingsley's lemma

Lemma 1.2 (Billingsley)**.** *Let (X, ρ) be a compact metric space, and let $A \in \mathcal{B}_X$. Suppose $\mu \in P(X)$ is such that for all $x \in A$, there exists a sequence of radii $\delta_1^x > \delta_2^x > \cdots \to 0$ such that $\mu(B(x, \delta_i^x)) \geq C(\delta_i^c)^\alpha$ for all i, where $C > 0$. Then $m_\alpha(A) < \infty$, and $\dim(A) \leq \alpha$.*

Proof. Let $\delta > 0$, and let $\mathcal{B}_\delta = \{B(x,r) : x \in A, \mu(B(x,r)) \geq Cr^\alpha, r \leq \delta\}$. For every such δ, $A \subseteq \bigcup \mathcal{B}_\delta$. Vitali's lemma provides disjoint $B(x_1, r_1), B(x_2, r_2), \dots \in B_\delta$ such that $A \subseteq \bigcup_i B(x_i, 4r_i)$. So

$$\mathcal{H}_{8\delta}^\alpha(A) \subseteq \sum_i (4r_i)^\alpha = 4^\alpha \sum_i r_i^\alpha \leq \frac{4^\alpha}{C} \sum_i \mu(B(x_i, r_i)) \leq \frac{4^\alpha}{C}.$$

So $m_\alpha(A) \leq 4^\alpha / C$. □

1.3 Local dimension

Definition 1.1. Let $\mu \in P(X)$. The **local dimension of μ at x** is

$$\dim(\mu, x) := \liminf_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

The **upper local dimension of μ at x** is

$$\overline{\dim}(\mu) = \inf\{\dim(A) : A \in \mathcal{B}_X, \mu(X \setminus A) = 0\},$$

and the **lower local dimension of μ at x** is

$$\underline{\dim}(\mu) = \inf\{\dim(A) : A \in \mathcal{B}_X, \mu(A) > 0\}.$$

We want to get out the biggest (i.e. supremum) exponent α we can choose so that $\mu(B(x, \delta)) \leq C\delta^\alpha$ for arbitrarily small balls.

Proposition 1.1. *Upper and lower local dimension have the following properties:*

1. $\overline{\dim}(\mu) = \operatorname{ess\,sup}_{x \sim \mu} \dim(\mu, x)$.
2. $\overline{\dim}(\mu) = \operatorname{ess\,inf}_{x \sim \mu} \dim(\mu, x)$,

where $x \sim \mu$ means that x is a random quantity drawn using the distribution μ.

Proof. For each of these, $\leq$ follows from Billingsley's lemma, and $\geq$ follows from the mass distribution principle. □

1.4 Frostman's lemma and weighted Hausdorff content

Lemma 1.3 (Frostman)**.** *Let (X, ρ) be a compact metric space. If $m_\alpha(X) > 0$, then there is a $\mu \in P(X)$ and a $C < \infty$ such that $\mu(B(x, r)) \leq Cr^\alpha$ for all x, r.*

Remark 1.2. We cannot just always take μ to be a normalized m_α because $m_\alpha(X)$ may be infinite.

We will prove this after introducing weighted Hausdorff measure.

Definition 1.2. Let $A \subseteq X$ and $\delta > 0$. The **weighted Hausdorff content** is

$$\mathcal{WH}_\delta^\alpha(A) := \inf \left\{ \sum_i c_I (\operatorname{diam}(E_i))^\alpha : \operatorname{diam}(E_i) \leq \delta, \mathbb{1}_A \leq \sum_i c_i \mathbb{1}_{E_i}, c_i \in (0, \infty) \right\} \leq \mathcal{H}_\delta^\alpha(A).$$

The **weighted Hausdorff measure** is

$$wm_\alpha(A) := \lim_{\delta \downarrow 0} \mathcal{WH}_\delta^\alpha(A).$$

Remark 1.3. From the definition, we see that $wm_\alpha \leq m_\alpha$.

Remark 1.4. The involved covering is not always just a covering of A. This is called a **fractional covering** of A.

Remark 1.5. Hausdorff measure is solving an optimization problem. Weighted Hasudorff measure is solving the relaxed[1] optimization problem.

Proposition 1.2. *Let A be compact. If $\mathcal{H}_\delta^\alpha(A) > 0$, then $\mathcal{WH}_{\delta/5}^\alpha(A) > 0$.*

Proof. Fix a fractional covering $\mathbb{1}_A \leq \sum_I c_i \mathbb{1}_{B_i}$ with $\operatorname{diam}(B_i) \leq 5$. By compactness, for all $t < 1$, there exists some M such that $t\mathbb{1}_A \leq \sum_{i=1}^M c_i \mathbb{1}_{B_i}$. We want to show that $\mathcal{H}_{5\delta}^\alpha(\{\sum_{i=1}^M c_i \mathbb{1} - B_i > t\}) \leq O(1/t) \sum_{i=1}^M a(\operatorname{diam}(B_i))^\alpha$. By perturbing, assume that the $c_i, t \in \mathbb{Q}_+$. Now clear denominators; assume $c_1, \dots, c_n, t \in \mathbb{N}$. By allowing duplicate balls B_i, we may assume that $c_i = 1$ for all i.

We now want to show that $\mathcal{H}_{5\delta}^\alpha(\{\sum_{i=1}^m \mathbb{1}_{B_i} > t\}) \leq \frac{O(1)}{t} \sum_i (\operatorname{diam}(B_i))^\alpha$. Let $\mathcal{B} = \{B_1, \dots, B_m\}$. Vitali's lemma gives disjoint $\tilde{B}_1, \dots, \tilde{B}_k \in \mathcal{B}$ such that $\bigcup_{j=1}^k \tilde{B}_j^{(3)} \supseteq \bigcup \mathcal{B}$. This means that $\tilde{\mathcal{B}} = \mathcal{B} \setminus \{\tilde{B}_1, \dots, \tilde{B}_k\}$ still covers A at least $t - 1$ times.

Now induct on t. For $t = 1/2$, we are done. For $t \in \mathbb{N} + 1/2$, assume we already know the statement for $t - 1$. We get

$$(t-1)\mathcal{H}_{5\delta}^\alpha(\underbrace{\{\sum \tilde{B} > t - 1\}}_{A}) \leq O(1) \sum_{B \in \mathcal{B}} (\operatorname{diam}(B))^\alpha.$$

[1] Relaxation is a notion from computer science.

Since $\bigcup_{j=1}^{k} \tilde{B}_j^{(3)} \supseteq \bigcup \mathcal{B} \supseteq A$, we also have

$$\mathcal{H}_{5\delta}^{\alpha}(A) \leq 3^{\alpha} \sum_{j=1}^{k} (\operatorname{diam}(\tilde{B}_j))^{\alpha}.$$

Now combine these two inequalities. □

We can now prove Frostman's lemma:

Proof. There exists a $\delta > 0$ such that $\mathcal{W}\mathcal{H}_{\delta}^{\alpha}(X) > 0$. Define for $f \in C(X)$:

$$p(f) := \inf \left\{ \sum_i c_i (\operatorname{diam}(B_i))^{\alpha} : f \leq \sum_i c_i \mathbb{1}_{B_i}, \operatorname{diam}(B_i) \leq \delta \right\}.$$

Check that $p(tf) = tp(f)$ for all $t > 0$, that $p(f+g) \leq p(f) + p(g)$, and that $p(\mathbb{1}_X) > 0$. By the Hahn-Banach theorem, we get a linear functional on the whole space. Now by Riesz representation, this is a measure. Take the total variation. □